Theorem - 26

The relation of Isomorphism in the set of all groups is an equivalence relation.

Proof Let $G = \{G_1, G_2, G_3, \ldots\}$ be the set of groups i.e. $G_1, G_2, G_3, \ldots$ are all groups.

We have to show that the relation of isomorphism which is denoted by "$\cong$" in the set G is an equivalence relation. That is we have to show that the relation is reflexive, symmetric and transitive.

Reflexive

Let G_i be any element of G.

~~We define~~ The identity mapping $f: G_i \to G_i$ be such that

$f_i(x) = x \ \forall x \in G_i$

Clearly f_i is one-one and onto.

Next if we take any two element x, y of G_i, then

$f_i(xy) = xy$ [$\because f$ is the identity mapping and G_i is a group]

$= f_i(x) f_i(y)$

So f_i is a homomorphism.

~~The~~ ~~The~~ Therefore f_i is an isomorphism.

Hence $G_i \cong G_i \ \forall G_i \in G$.

So the relation '$\cong$' is reflexive.

Symmetric

Let $G_i, G_j \in G$ and $G_i \cong G_j$

So there exists an isomorphism $f: G_i \to G_j$.

Then $f^{-1}: G_j \to G_i$ is also an isomorphism. [see theorem-23]

Therefore $G_j \cong G_i$

Hence the relation '$\cong$' is symmetric.

Transitive

Let $G_i, G_j, G_K \in G$ and $G_i \cong G_j$, $G_j \cong G_K$

Then there exist two isomorphisms f and g such that

$f: G_i \to G_j$ and $g: G_j \to G_K$.

Now from the theorem 24 we get ~~the~~ the composite mapping $gf: G_i \to G_K$ is also an isomorphism.

So $G_i \cong G_K$

Hence the relation '$\cong$' is transitive.

Hence the relation '$\cong$' is an equivalence relation in the set of groups G.

Theorem-27

Suppose G is a group and N is a normal subgroup of G. If we define a mapping f from G to G/N as $f(x) = Nx \ \forall \ x \in G$, then f is a homomorphism of G onto G/N with $\text{Ker} f = N$.

Proof We ~~o~~ first show that f is an onto mapping. Let Nx be any element of G/H. Then Nx has a pre-image x in G ~~oo~~ since $f(x) = Nx$. Therefore f is an onto mapping.

Next we show that f is a homomorphism.
Let x, y be any two elements of G, then $f(x) = Nx$ and $f(y) = Ny$ ~~and f(xy)=~~

Now $x, y \in G \Rightarrow xy \in G$.

So $f(xy) = Nxy = (Nx)(Ny) = f(x) f(y)$

Therefore f is a homomorphism.

Hence f is a homomorphism of G onto G/N.

Next let K be the Kernel of this homomorphism f. We know that the identity element of the quotient group G/N is N.

Therefore $K = \{ y \in G : f(y) = N \}$

Next we have to prove ~~xxx~~ $K = N$.

~~Let We oo~~

We take any $k \in K$.

Since K is the kernel, so $f(k) = N$, as N is the identity of G/N.

Again from the definition $f(k) = Nk$

Therefore $Nk = N$

which implies $k \in N$.

So $k \in K \Rightarrow k \in N$, therefore $K \subseteq N$. -- (1)

Again n be any element of N.

So $f(n) = Nn = N$

Therefore $n \in K$

Which implies $N \subseteq K$ -- (2)

From (1) and (2), we get, $K = N$

Therefore f is a homomorphism of G onto G/N with $\text{Ker} f = N$.

Note

Every quotient group of a group is a homomorphic image of the group. The mapping $f: G \to G/N$ such that $f(x) = Nx \ \forall x \in G$ is called a natural or canonical homomorphism of G onto G/N.

Theorem-28 (Cayley's Theorem)

Every group is isomorphic to a permutation group.

Proof Let $(G, \circ)$ be a group and $A(G)$ be the group of all permutations of G. We shall show that G is isomorphic to a subgroup of $A(G)$.

Let us take an element a of G.

We define a mapping $f_a : G \to G$ by

$f_a(g) = a \circ g \ \forall g \in G$.

We shall first show that f_a is a bijective

Let g_1, g_2 be any two elements of G and

$f_a(g_1) = f_a(g_2)$

$\Rightarrow a \circ g_1 = a \circ g_2$

$\Rightarrow g_1 = g_2$ [By left cancellation law, as G is a group]

So f_a is one-one.

Again let g be any element of G (Codomain)

Since $a \in G$, so $a^{-1} \in G$ and $a^{-1} \circ g \in G$.

Now $f_a(a^{-1} \circ g) = a \circ (a^{-1} \circ g) = (a \circ a^{-1}) \circ g$ [since G is a group]

$= e \circ g$ [e is the identity element of G]

$= g$

So each element of $g \in G$ (Codomain) there exists an ~~element~~ pre-image $a^{-1} \circ g$ in G (domain)

Therefore f_a is a bijective mapping from G to G.

So f_a is a permutation, so $f_a \in A(G)$

Next we take two elements a, b of G.

Now $f_{a \circ b}(x) = (a \circ b) \circ x = a \circ (b \circ x) = f_a(b \circ x)$

$= f_a(f_b(x)) = (f_a f_b)(x) \ \forall x \in G$.

$\therefore f_{a \circ b} = f_a f_b$ ---- (1)

Next we define a mapping $\phi : G \to A(G)$ by $\phi(a) = f_a \ \forall$ all $a \in G$.

From (1) we can conclude that ϕ is a homomorphism

Now for any two elements a, b of G

$$\phi(a) = \phi(b) \Rightarrow f_a = f_b \quad \text{so } f_a(e) = f_b(e) \; [\because e \in G]$$

$$\Rightarrow a \circ e = b \circ e \Rightarrow a = b$$

So ϕ is one-one.

~~Next we show that $\phi(G)$ is a subgroup of $A(G)$;~~
~~where $\phi(G) = \{f_a : a \in G\}$ is a subset of $A(G)$~~
~~Since $e \in G$, and~~
~~Now $\phi(e) = f_e \in \phi(G)$ [where $f_e(x) = e \circ x = x, \forall x \in G$]~~
~~Therefore $\phi(G)$ is non-empty.~~

Since ϕ is a homomorphism, so $\phi(G)$ is a subgroup of $A(G)$ [See theorem-18 and G is a subgroup of G] and being a subgroup of $A(G)$, $\phi(G)$ is also a permutation group.

Finally we can conclude that $\phi : G \to A(G)$ is an isomorphism.

Hence every group is isomorphic to a permutation group.

<u>Note</u> If G is a finite group of order n, then G is isomorphic to a subgroup of the symmetric group S_n.

<u>First Isomorphism Theorem</u>

<u>Theorem-29</u>.

Every homomorphic image of a group G is isomorphic to some quotient group of G.

<u>Proof</u> Let G_1 be the homomorphic image of a group G for the homomorphism f. Clearly $f : G \to G_1$ is an onto homomorphism.

Let K be the Kernel of this homomorphism. Then K is a normal subgroup of G. Next we shall prove that $G/K \cong G_1$.

If $x \in G$, then $Kx \in G/K$ and $f(x) \in G_1$.

Next we define the mapping $\phi : G/K \to G_1$ such that $\phi(Kx) = f(x) \ \forall \ x \in G$.

We shall first show that ϕ is well-defined.

Let $x, y \in G$ and $Kx = Ky$

$$\Rightarrow xy^{-1} \in K.$$

$$\Rightarrow f(xy^{-1}) = e_1 \ [\text{The identity of } G_1]$$

$$\Rightarrow f(x) f(y^{-1}) = e_1$$

$$\Rightarrow f(x)\{f(y)\}^{-1} = e_1 \quad [\because f(y^{-1}) = \{f(y)\}^{-1}]$$

$$\Rightarrow f(x)\{f(y)\}^{-1} f(y) = e_1 f(y)$$

$$\Rightarrow f(x) e_1 = f(y)$$

$$\Rightarrow f(x) = f(y)$$

$$\Rightarrow \phi(Kx) = \phi(Ky)$$

Therefore ϕ is well-defined.

Next we show that ϕ is one-one.

Let $\phi(Ka) = \phi(Kb)$

$$\Rightarrow f(a) = f(b)$$

$$\Rightarrow f(a)\{f(b)\}^{-1} = f(b)\{f(b)\}^{-1}$$

$$\Rightarrow f(a) f(b^{-1}) = e_1 \quad [\because \{f(b)\}^{-1} = f(b^{-1})]$$

$$\Rightarrow f(ab^{-1}) = e_1$$

$$\Rightarrow ab^{-1} \in K \quad [\text{Since } K \text{ is the Kernel of the homomorphism}]$$

$$\Rightarrow Ka = Kb$$

So ϕ is one-one.

Next take y be any element of G_1. Then there exists an $c \in G$ such that $f(c) = y$ [Since f is onto mapping from G to G_1]

Now $Kc \in G/K$ and we can write

$$\phi(Kc) = f(c) = y.$$

$\therefore$ ϕ is onto.

Finally we have $\phi\{(Ka)(Kb)\} = \phi(Kab) = f(ab)$

$$= f(a) f(b) = \phi(Ka)\phi(Kb)$$

$\therefore$ ϕ is an isomorphism from G/K to K_1

Therefore $G/K \cong G_1$

Note: This theorem is also known as the Fundamental theorem of homomorphism.

Second Theorem of Isomorphism

Theorem-30

Let H and K be subgroups of a group G with K normal in G. Then $\dfrac{HK}{K} \cong \dfrac{H}{H \cap K}$

Proof Here H is a subgroup of G and K is a normal subgroup of G. So $H \cap K$ is a normal subgroup of H and consequently $\dfrac{H}{H\cap K}$ is a quotient group.

Again HK is a subgroup of G and $K \subseteq HK$. Now K is normal in G, so K is also a normal subgroup of HK.

Hence $\dfrac{HK}{K}$ is also a quotient group.

Next we consider the mapping $f: H \to \dfrac{HK}{K}$ defined by

$$f(x) = Kx$$

Now $x_1 = x_2 \Rightarrow Kx_1 = Kx_2 \Rightarrow f(x_1) = f(x_2)$

Therefore f is well-defined.

Again $f(xy) = Kxy = (Kx)(Ky) = f(x)\, f(y)$

Therefore f is a homomorphism.

~~Again~~ Next Kx be any element of $\frac{HK}{K}$.

Then $x \in HK$.

So $x = hk$ for some $h \in H$ and $k \in K$

Again K is a normal subgroup of G and H is a subgroup of G, so $HK = KH$ [see theorem 10]

So there exist $k_1 \in K$ and $h_1 \in H$ such that

$hk = k_1 h_1$

Now we have $f(h_1) = Kh_1 = (Kk_1)h_1 \left[\begin{array}{l} \because k_1 \in K \\ \therefore Kk_1 = K \end{array}\right]$

$= K(k_1 h_1) = K(hk) = Kx$

So for $Kx \in \frac{HK}{K}$, there exists a $h_1 \in H$ such that $f(h_1) = Kx$

Therefore f is onto

Hence f is a homomorphism of H onto $\frac{HK}{K}$.

Next we try to show kernel of f is $H \cap K$, which is a subset of H.

The identity element of the quotient group $\frac{HK}{K}$ is K

~~Let $h \in H$~~

An element h of H will be an element of Kernel of f if $f(h) = K$

$\Rightarrow Kh = K$

$\Rightarrow h \in K$

$\Rightarrow h \in H \cap K$ [$\because h \in H$ and $h \in K$]

So $Ker f \subseteq H \cap K$

Again if $h \in H \cap K$, then $h \in H$ and $h \in K$

Now $f(h) = Kh = K$

So $h \in H \cap K \Rightarrow h \in Ker f$

Therefore $H \cap K \subseteq Ker f$

Hence $Ker f = H \cap K$.

Then by the fundamental theorem on homomorphism of groups, we have $HK/K \cong \dfrac{H}{H \cap K}$.

Third theorem of Isomorphism

~~Theorem-31~~

~~If H and K are two normal subgroups of a group G, such that $H \subseteq K$, then $G/K \cong \dfrac{(G/H)}{(K/H)}$.~~

Theorem-31 If H and K are two normal subgroups of a group G such that $H \subseteq K$, then K/H is a normal subgroup of G/H. ~~Conversely, if K/H is a normal subgroup of G/H~~

Proof

Here H is a normal subgroup of G and $H \subseteq K$, where K is also a normal subgroup of G. Therefore H is also a normal subgroup of K and so K/H is a quotient group.

Obviously K/H is non-empty.

Let Ha be an element of K/H, where $a \in K$.

But $a \in K \Rightarrow a \in G$. Therefore Ha is also an element of G/H.

So K/H is a non-empty subset of G/H

Next we take $Ha_1, Ha_2 \in \frac{K}{H}$.

Now $(Ha_1)(Ha_2)^{-1} = (Ha_1)(Ha_2^{-1}) = Ha_1a_2^{-1} \in \frac{K}{H}$ $[\because (Hg)^{-1} = Hg^{-1}]$ $\left[\because a_1 a_2 \in K \Rightarrow a_1 a_2^{-1} \in K\right]$

Therefore $\frac{K}{H}$ is a subgroup of G/H.

Next we show that K/H is a normal subgroup of G/H. Let Hg and Hk be any two elements of G/H and K/H respectively, where $g \in G$ and $k \in K$.

Now $(Hg)(Hk)(Hg)^{-1} = (Hg)(Hk)(Hg^{-1})$

$= Hgkg^{-1}$ $\left[\because H \text{ is normal, so } (Ha)(Hb) = Hab\right]$.

Since K is a normal subgroup of G, so $gkg^{-1} \in K$. Hence $Hgkg^{-1} \in K/H$.

Therefore K/H is a normal subgroup of G/H.

Third Theorem of Isomorphism

Theorem-32

If H and K are two normal subgroups of a group G, such that $H \subseteq K$, then

$$G/K \cong \frac{(G/H)}{(K/H)}$$

proof

Since H is a normal of G and K is a normal subgroup of G containing H, then by theorem-31, we conclude that the quotient group K/H is a normal subgroup of the quotient group G/H.

Therefore $\frac{(G/H)}{(K/H)}$ is ~~atoa~~ also a quotient group.

Next we consider the mapping $f : G/H \to G/K$ defined by $f(Hg) = Kg$, $g \in G$.

We shall first show that f is well-defined.

Let $Hg_1 = Hg_2$, where $g_1, g_2 \in G$.

So $g_1 g_2^{-1} \in H$

$\Rightarrow g_1 g_2^{-1} \in K$ $[\because H \subseteq K]$

$\Rightarrow Kg_1 = Kg_2$

$\Rightarrow f(Hg_1) = f(Hg_2)$

So f is well-defined.

Again $f[(Hg_1)(Hg_2)] = f(Hg_1g_2) = Kg_1g_2$ $[\because g_1g_2 \in G]$

$= (Kg_1)(Kg_2) = f(Hg_1)f(Hg_2)$

So f is a homomorphism from G/H to G/K.

Next we shall show that f is onto.

~~Let~~ For any $Ky \in G/K$ there exists a coset $Hy \in G/H$ such that $f(Hy) = Ky$

$\therefore$ f is a homomorphism of G/H onto G/K.

Next we shall show that $\text{Ker} f = K/H$.

We know that the identity element of G/K is K.

Let $Hx \in G/H$, ~~be an element of Kerf, then~~

then $Hx \in \text{Ker} f$

$\Leftrightarrow f(Hx) = K$ [where $\Leftrightarrow$ means implies and implied by]

$\Leftrightarrow Kx = K$

$\Leftrightarrow x \in K$

$\Leftrightarrow \; Hx \in K/H$

Therefore $\operatorname{Ker} f = K/H$

Hence f is a homomorphism of G/H onto G/K with Kernel K/H.

So by the fundamental theorem on homomorphism of groups, we get $G/K \cong \dfrac{(G/H)}{(K/H)}$.